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AN E[illegible]ATION OF STONE'S SOLUTION FOR A SLIGHTLY YAWING SUPERSONIC CONE

G. R. W. Young

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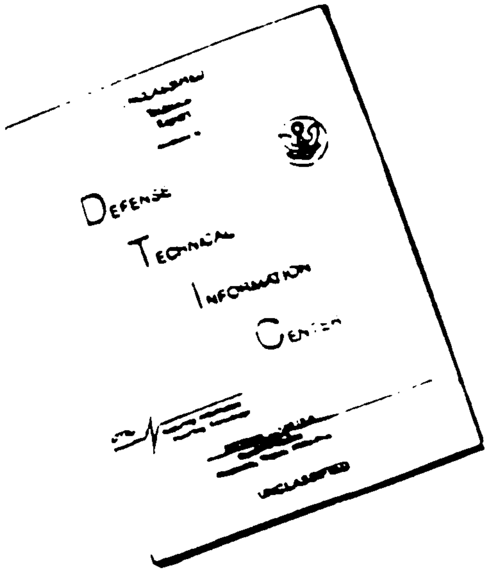

PROJECT RAND

RESEARCH MEMORANDUM

AN ELUCIDATION OF STONE'S SOLUTION FOR A SLIGHTLY YAWING SUPERSONIC CONE

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AN ELUCIDATION OF STONE'S SOLUTION* FOR A SLIGHTLY YAWING SUPERSONIC CONE

By

G. B. W. Young

Introduction

Stone's solution for a slightly yawing supersonic cone takes advantage of the fact that the differential equations (equation of motion, equation of continuity, etc.) to be solved for flow over a slightly yawing cone are identical with the differential equations that must be solved for flow over a non-yaw cone with small but finite w-components of velocity caused by external forces. This identity implies that any solution obtained for the second problem must be applicable to the first flow problem provided the boundary conditions were properly chosen.

With the aid of the known solution (presented by Taylor and Maccoll) for flow over a non-yaw cone with zero w-components of velocity, the solution for the flow field about a non-yaw cone with small w-components of velocity can be approximated as follows. Since the latter flow approaches the former flow field as the finite w-components of velocity tend to zero, one can postulate that for small w-components of velocity, the solution for the flow field with finite w-components of velocity is equal to the solution for the flow field with zero w-components of velocity plus a deviation resulting from the presence of the finite but small w-components of velocity.

The contributions of Stone, then, are a proper choice of the boundary conditions and the evaluation of the deviations mentioned. Typical flow fields

*Stone, A. H., "The Aerodynamics of a Slightly Yawing Supersonic Cone", NDRC No. A-358, OSRD No. 6306, July 10, 1945.

for flow over a non-yaw cone with zero w-components of velocity, for flow over a non-yaw cone with finite w-components of velocity and for flow over a yawing cone are presented in Figs. 1, 2 and 3 at a Mach number of 3.1617 for a cone with a half angle of 30 degrees (for $\phi = 0^{\circ}, 180^{\circ}$).

Thus Stone's solution includes the following three basic steps:

(a) The solution of Taylor and Maccoll for the fluid properties of non-yaw motion is assumed to be correct. The differential equations describing non-

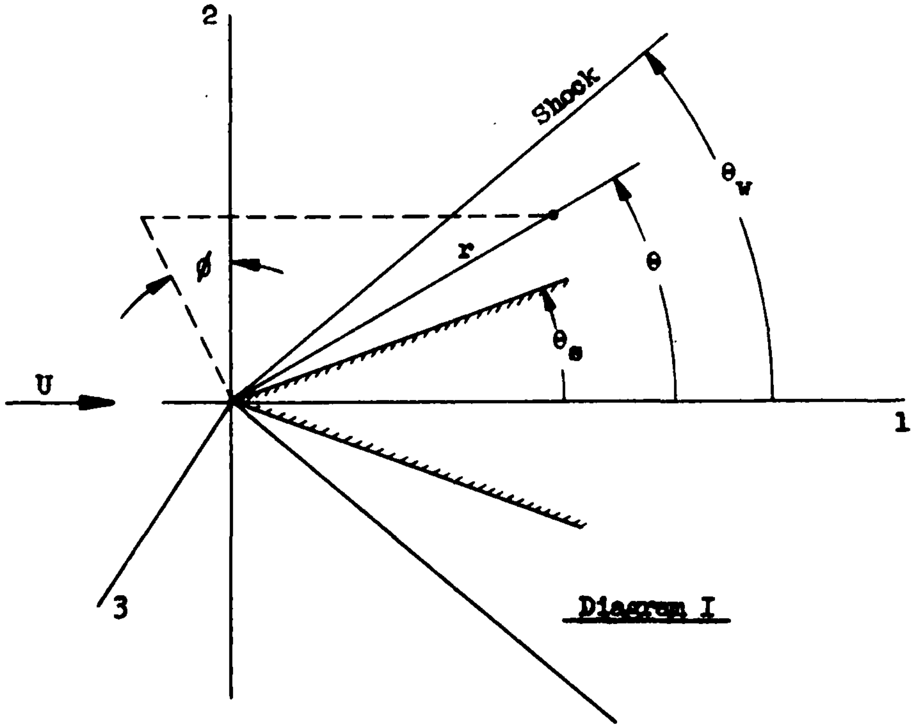

Diagram I

yaw motion with zero w-components of velocity are:

1. The equations of motion

$$\bar{u}' = \bar{v}$$

$$\frac{\bar{P}}{\rho} = - \bar{u}' (\bar{u} + \bar{u}'')$$

2. The continuity equation

$$\bar{u}'' + \left[\cot\theta + (\ln\bar{\rho})'\right]\bar{u}' + 2\bar{u} = 0$$

3. The equation of state

$$\ln\bar{P} - \gamma\ln\bar{\rho} = \text{contant.}$$

The boundary conditions for non-yaw motion with zero w-components of velocity, referring to Diagram I, are:

$$\text{at}\quad \theta = \theta_s, \qquad \bar{v} = \bar{u}' = 0$$

$$\text{at}\quad \theta = \theta_w, \qquad \bar{u} = U\cos\theta$$

$$U\rho_1\sin\theta + \bar{\rho}\bar{v} = 0$$

$$\bar{P} - P_1 = U\rho_1\sin\theta\,(\bar{v} + U\sin\theta)$$

$$U^2\sin^2\theta = \frac{(\gamma-1)P_1 + (\gamma+1)\bar{P}}{2\rho_1}$$

where the bars denote the Taylor and Maccoll's or non-yaw solution with zero w-components of velocity, the subscripts 1 denote free stream conditions and the primes denote differentiation with respect to θ.

(b) A problem of flow about a non-yaw cone with small w-components of velocity is set up with boundary conditions that will have physical meaning upon application to the flow over a yawing cone. The differential equations describing non-yaw motion with small w-components of velocity in the region between θ_w to θ_s are:

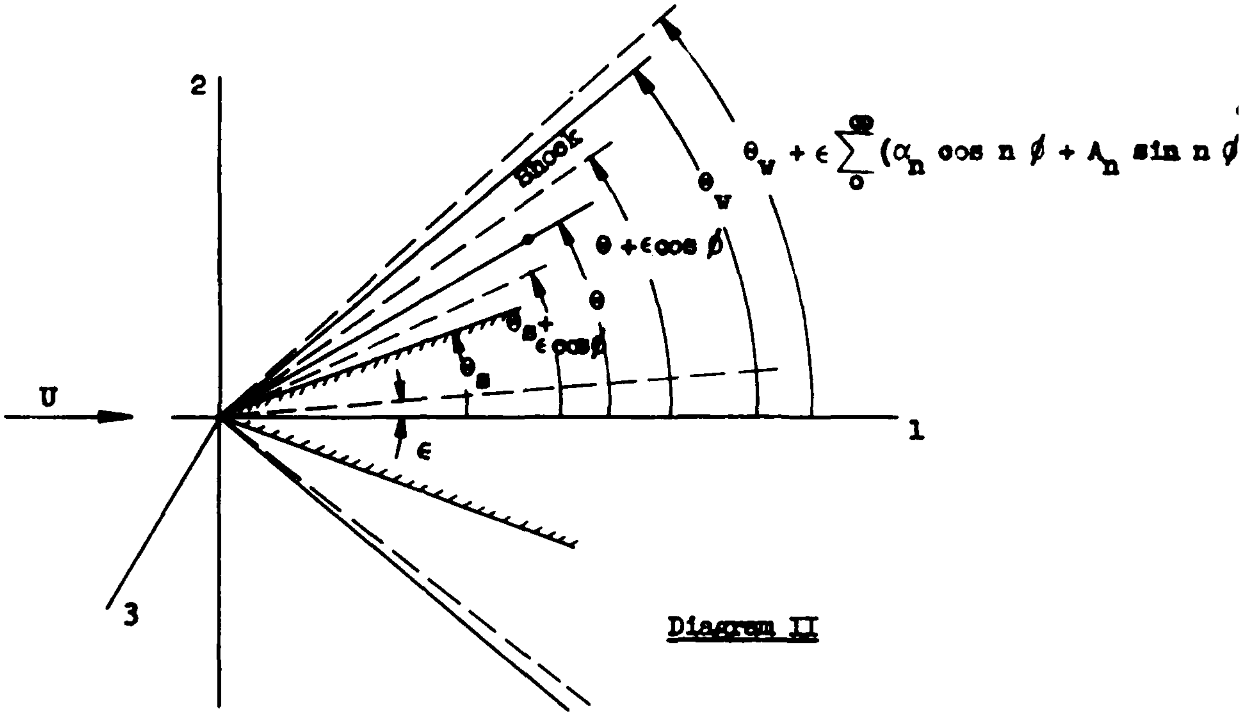

Diagram II

1. The equations of motion

$$u' = v$$

$$\frac{p'}{\rho} = - u' (u + u'')$$

$$\frac{1}{\rho \sin \theta} \frac{\partial p}{\partial \phi} = - (uw + vw \cot \theta)$$

2. The continuity equation

$$u' (\ln \rho)' + 2u + v \cot \theta + v' + v \frac{\csc \theta}{\rho} \frac{\partial \rho}{\partial \phi} = 0$$

3. The equation of state

$$\ln P - \gamma \ln \rho = f(\epsilon, \phi)$$

Referring to Diagram II, ϵ is the yaw angle of the corresponding problem of flow over a yawing cone and $\theta_v + \sum_0^\infty (\alpha_n \cos n\phi + A_n \sin n\phi)$, the shock front. The boundary conditions for non-yaw motion with small v-components of velocity are obtained in the following manner. At the cone surface $(\theta = \theta_s)$ the values of v_{θ_s} and v'_{θ_s} $\left[= \left(\frac{\partial v}{\partial \theta}\right)_{\theta_s}\right]$ must satisfy the relation

$$v_{\theta_s} + v'_{\theta_s} \epsilon \cos \phi = 0$$

This expression states that the normal velocities on a cone surface of $\theta_s + \epsilon \cos \phi$ are equal to zero. This cone surface $(\theta_s + \epsilon \cos \phi)$ describes a cone yawed at an angle of ϵ with respect to the free stream axis. Similarly at the shock wave surface $(\theta = \theta_v)$, the fluid properties (u, v, etc.) and their derivatives (with respect to θ) are related in such a manner that the Rankine-Hugoniot equations are satisfied at the surface given by

$$\theta_v + \epsilon \sum_0^\infty (\alpha_n \cos n\phi + A_n \sin n\phi).$$

This surface will be shown later to be the shock wave surface about a slightly yawing cone.

Now, the v-components of velocity that are of concern, in the present problem, are equivalent to those caused by a small yaw angle in the flow about a yawing cone. Since v and ϵ are of the same order of magnitude one can postulate that the fluid properties about a non-yaw cone with small v-components of velocity are related to the fluid properties about a non-yaw cone with zero v-components of velocity as follows:

$$u_\theta = \bar{u}_\theta + \epsilon f_{11} + \epsilon^2 f_{12} + \epsilon^3 f_{13} \ldots.$$

$$v_\theta = \bar{v}_\theta + \epsilon f_{21} + \epsilon^2 f_{22} + \epsilon^3 f_{23} \; \cdots\cdot$$

$$w_\theta = \bar{w}_\theta + \epsilon f_{31} + \epsilon^2 f_{32} + \epsilon^3 f_{33} \; \cdots\cdot$$

$$P_\theta = \bar{P}_\theta + \epsilon f_{41} + \epsilon^2 f_{42} + \epsilon^3 f_{43} \; \cdots\cdot$$

$$\rho_\theta = \bar{\rho}_\theta + \epsilon f_{51} + \epsilon^2 f_{52} + \epsilon^3 f_{53} \; \cdots\cdot$$

where the f's are function of ϕ. The functions f's are analytic and periodic and thus could be expanded into Fourier Series. Neglecting the higher order ϵ's (ϵ^2, ϵ^3, etc.), then

$$u_\theta = \bar{u}_\theta + \epsilon \sum_{n=0}^{\infty} (x_n \cos n\phi + X_n \sin n\phi)$$

$$v_\theta = \bar{v}_\theta + \epsilon \sum_{n=0}^{\infty} (y_n \cos n\phi + Y_n \sin n\phi)$$

$$w_\theta = \epsilon \sum_{n=0}^{\infty} (Z_n \cos n\phi + z_n \sin n\phi)$$

$$P_\theta = \bar{P}_\theta + \epsilon \sum_{n=0}^{\infty} (\eta_n \cos n\phi + H_n \sin n\phi)$$

$$\rho_\theta = \bar{\rho}_\theta + \epsilon \sum_{n=0}^{\infty} (\xi_n \cos n\phi + \Xi_n \sin n\phi)$$

The coefficients are evaluated by substituting these expressions into the differential equations and the boundary condition relationships. Thus the fluid properties of flow about a non-yaw cone with finite but small w-components of velocity are obtained. This flow field will be designated as the imaginary flow field because it has no physical meaning. It only

serves as an intermediate step in the determination of flow about a yawing cone.

(c) The last step of Stone's solution is the application of the results obtained for the flow past a non-yaw cone with small v-components of velocity to the problem of flow about a yawing cone. It has been mentioned that the differential

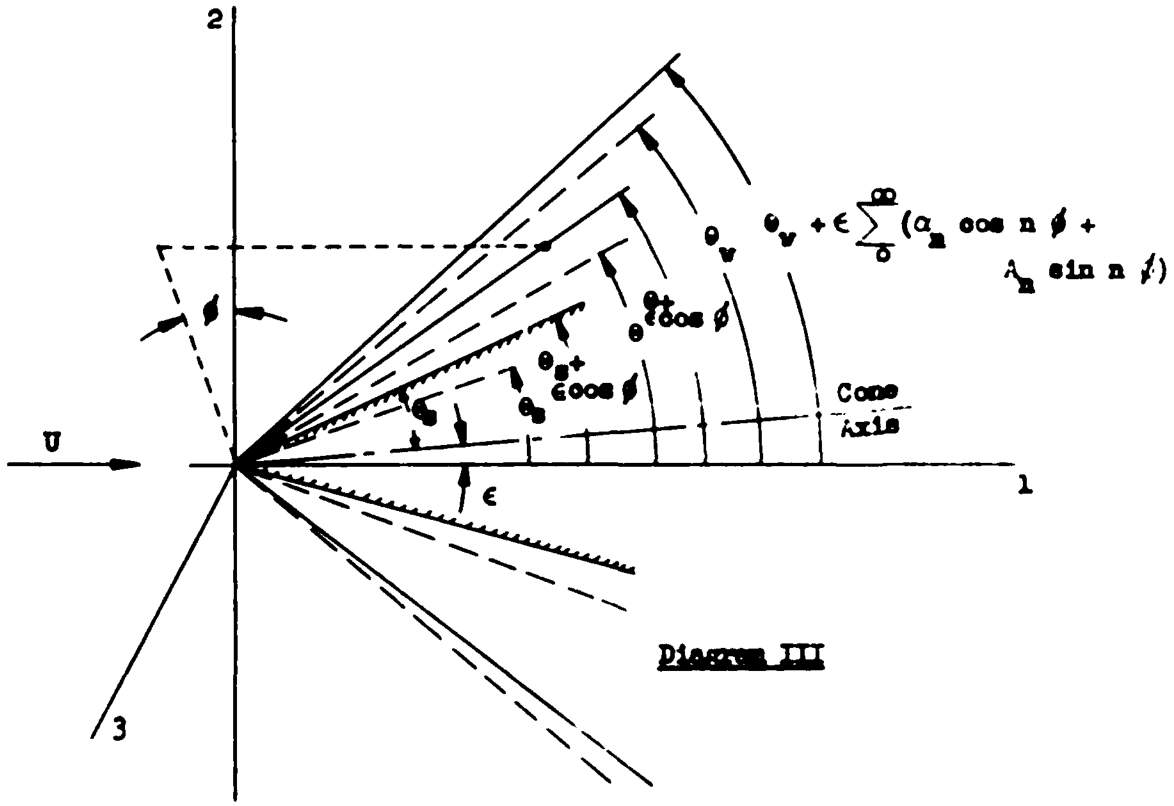

Diagram III

equations describing these two problems are identical. Therefore the results obtained for the fluid properties in Step (b) may be used in the region between θ_w and θ_s in Diagram III. However, the direct transfer of the results has no physical meaning due to the manner in which the boundary conditions were chosen in Step (b); that is, the fluid properties, u_θ, v_θ, w_θ, P_θ, and ρ_θ are not the fluid properties for the flow about a yawing cone.

The boundary conditions require that at $\theta_s + \epsilon \cos \phi$, the yawing cone surface, the normal velocities be zero and at $\theta_v + \epsilon \sum_{n=0}^{\infty} (\alpha_n \cos n \phi + A_n \sin n \phi)$ the shock wave surface, the Rankine-Hugoniot conditions be satisfied. These boundary conditions can only be satisfied when the fluid properties are evaluated as follows:

$$u_r = u(\theta + \epsilon \cos \phi) = u_\theta + u'_\theta \; \epsilon \cos \phi$$

$$v_r = v(\theta + \epsilon \cos \phi) = v_\theta + v'_\theta \; \epsilon \cos \phi$$

$$w_r = w(\theta + \epsilon \cos \phi) = w_\theta + w'_\theta \; \epsilon \cos \phi$$

$$p_r = p(\theta + \epsilon \cos \phi) = p_\theta + p'_\theta \; \epsilon \cos \phi$$

$$\rho_r = \rho(\theta + \epsilon \cos \phi) = \rho_\theta + \rho'_\theta \; \epsilon \cos \phi$$

In these expressions, the variables, u_θ, u'_θ, v_θ, v'_θ, etc., satisfy the mutual differential equations, thus u_r, v_r, etc., also satisfy the mutual differential equations. Due to the manner in which the boundary conditions were chosen in Step (b), the fluid properties u_r, v_r, etc., also satisfy the boundary conditions specified by the problem of flow past a supersonic yawing cone.

In brief, then, the method of Stone consists of the evaluation of the deviations of the fluid properties for non-yaw motion with small w's from the fluid properties for non-yaw motion with zero w's and the calculation of the fluid properties for yawing motion by the rotation of each conical surface as shown in Diagrams II and III.

Analysis

The general equations of steady motion for three-dimensional flow in spherical coordinates, neglecting the effects of viscosity, are

$$u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial u}{\partial \phi} + \frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{1}{r}\left(v^2 + w^2\right) = 0$$

$$u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial v}{\partial \phi} + \frac{1}{\rho r}\frac{\partial P}{\partial \theta} + \frac{1}{r}\left(uv - w^2\cot\theta\right) = 0 \qquad (1)$$

$$u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial w}{\partial \phi} + \frac{1}{\rho\, r\sin\theta}\frac{\partial P}{\partial \phi} + \frac{1}{r}\left(uw + vw\cot\theta\right) = 0$$

The equation of continuity is

$$\frac{\partial}{\partial r}\left(\rho\, r^2 u\sin\theta\right) + \frac{\partial}{\partial \theta}\left(\rho\, r\, v\sin\theta\right) + \frac{\partial}{\partial \phi}\left(\rho\, r\, w\right) = 0 \qquad (2)$$

The equation of state for steady motion and upon the postulation that the flow is adiabatic along each streamline is

$$\frac{dP}{P} = \gamma\frac{d\rho}{\rho}$$

where the streamline direction is given by

$$\frac{dr}{u} = \frac{r\,d\theta}{v} = \frac{r\sin\theta\,d\phi}{w}$$

Hence

$$\frac{\gamma P}{\rho} = \frac{dP}{d\rho} = \frac{\frac{\partial P}{\partial r}dr + \frac{\partial P}{\partial \theta}d\theta + \frac{\partial P}{\partial \phi}d\phi}{\frac{\partial \rho}{\partial r}dr + \frac{\partial \rho}{\partial \theta}d\theta + \frac{\partial \rho}{\partial \phi}d\phi}$$

$$= \frac{u \frac{\partial P}{\partial r} + \frac{v}{r} \frac{\partial P}{\partial \theta} + \frac{w}{r \sin \theta} \frac{\partial P}{\partial \phi}}{u \frac{\partial \rho}{\partial r} + \frac{v}{r} \frac{\partial \rho}{\partial \theta} + \frac{w}{r \sin \theta} \frac{\partial \rho}{\partial \phi}} \tag{3}$$

Equations (1), (2) and (3) must be satisfied for any three dimensional, steady state fluid motion where viscosity can be neglected.

For non-yaw flow about a cone these equations were simplified and solved by Taylor and Maccoll for the region between the shock wave and the cone. The simplification was made by postulating that the fluid properties do not change along any radius element emanating from the vertex of the cone and that the w-component of velocity is zero. Thus the equations of motion (1) become,

$$\bar{u}' = \bar{v} \tag{4}$$

$$\frac{\bar{P}'}{\bar{\rho}} = - \bar{u}' \left(\bar{u} + \bar{u}''\right) \tag{5}$$

$$0 = 0 \tag{6}$$

the continuity equation becomes

$$\bar{u}'' + \left[\cot \theta + (\ln \bar{\rho})'\right] \bar{u}' + 2\bar{u} = 0 \tag{7}$$

and the equation of state becomes

$$\frac{\bar{P}}{\bar{\rho}^{\gamma}} = \text{constant} \tag{8a}$$

or

$$\ln \bar{P} - \gamma \ln \bar{\rho} = \text{constant} \tag{8b}$$

where the barred letters, $\bar{u}$, $\bar{v}$, etc. are used to refer to the non-yaw case, and the primes denote differentiation with respect to θ. The boundary conditions employed by Taylor and Maccoll are:

$$\text{when} \quad \theta = \theta_s, \quad \bar{v} = \bar{u}' = 0$$

$$\text{when} \quad \theta = \theta_w, \quad \bar{u} = U \cos \theta$$

$$U\rho_1 \sin \theta + \bar{\rho}\bar{v} = 0 \tag{9}$$

$$\bar{P} - P_1 = U\rho_1 \sin \theta \, (\bar{v} + U \sin \theta)$$

$$U^2 \sin^2 \theta = \left[(\gamma - 1) P_1 + (\gamma + 1) \bar{P} \right] \frac{1}{2\rho_1}$$

where θ_s denotes the cone surface; θ_w, denotes the shock wave; U, the free stream velocity, and the subscripts 1 denote free stream conditions.

In the case of flow past a yawed cone the postulation of the non-variance of the fluid properties along r may still be used, but the w-component of the velocity cannot be assumed zero. Stone, following the work of Karush and Critchfield, postulated that w and the variation of fluid properties in the w (or ϕ) direction are of the order of magnitude of the yaw angle, ϵ, and that for slightly yawing cones (small ϵ), the terms in the above equations of the order of magnitude of ϵ^2 or higher can be neglected. The resulting equations are:

Equations of motion

$$u' = v \tag{10}$$

$$\frac{P'}{\rho} = - u' (u + u'') \tag{11}$$

$$\frac{1}{\rho \sin \theta} \frac{\partial P}{\partial \phi} = - (uw + vw \cot \theta) \tag{12}$$

The continuity equation,

$$u' (\ln \rho)' + 2u + v \cot \theta + v' + w \frac{\csc \theta}{\rho} \frac{\partial \rho}{\partial \phi} + \frac{\partial w}{\partial \phi} \csc = 0 \tag{13}$$

The equation of state,

$$\frac{\gamma P}{\rho} = \frac{\frac{\partial P}{\partial \theta}}{\frac{\partial \rho}{\partial \theta}}$$

or $$\frac{1}{P} \frac{\partial P}{\partial \theta} = \frac{\gamma}{\rho} \frac{\partial \rho}{\partial \theta}$$

integrating with θ,

$$\ln P - \gamma \ln \rho = f(\epsilon, \phi) \tag{14}$$

(notice that f is independent of θ)

The fluid properties in the region between the shock wave and the cone for a slightly yawing cone must satisfy these equations (Eqs. 10 to 14). At this point one must remember that these equations do not demand the existence of a yawing cone. These equations can also describe the flow about a non-yaw cone with finite w-component of velocity caused by external forces. The magnitude of w cannot, however, exceed the w resulting from flow past a slightly yawing cone.

Due to the complexity of these equations, Stone, also Karush and Critchfield, took advantage of the solution of Taylor and Maccoll for non-yaw motion and obtained a solution to these equations, which yield an imaginary flow field about a non-yaw cone with finite w's. Then the imaginary flow field is rotated to give the approximate flow field of fluid motion past a yawed cone.

The imaginary flow field is obtained in the following manner. Since the fluid properties of the imaginary flow field must degenerate to the solution of Taylor and Maccoll as w approaches zero, the equivalent of ϵ approaches zero, and the deviation from the solution of Taylor and Maccoll is small for small w or ϵ, Stone postulated that

$$u = \bar{u} + \epsilon f_1$$

$$v = \bar{v} + \epsilon f_2$$

$$w = \bar{w} + \epsilon f_3$$

$$P = \bar{P} + \epsilon f_4$$

$$\rho = \bar{\rho} + \epsilon f_5$$

For small w or ϵ, the functions, f's, are only a function of ϕ and are periodic and thus can be expanded into Fourier's Series (notice that terms of ϵ^2 and higher are neglected). That is

$$u = \bar{u} + \epsilon \sum_{n=0}^{\infty} (x_n \cos n\phi + X_n \sin n\phi)$$

$$v = \bar{v} + \epsilon \sum_{n=0}^{\infty} (y_n \cos n\phi + Y_n \sin n\phi)$$

$$w = 0 + \epsilon \sum_{n=0}^{\infty} (Z_n \cos n\phi + z_n \sin n\phi)$$

$$P = \bar{P} + \epsilon \sum_{n=0}^{\infty} (\eta_n \cos n\phi + H_n \sin n\phi)$$

$$\rho = \bar{\rho} + \epsilon \sum_{n=0}^{\infty} (\xi_n \cos n\phi + \Xi_n \sin n\phi)$$

Since the flow field about a non-yaw cone is symmetrical with the plane ($\phi = 0$, $\theta = 0$) and the magnitude of w (or ϵ) is small, the imaginary flow field is expected to be symmetrical. Thus

$$
\begin{aligned}
u &= \bar{u} + \epsilon \sum_{n=0}^{\infty} (x_n \cos n\phi) \\
v &= \bar{v} + \epsilon \sum_{n=0}^{\infty} (y_n \cos n\phi) \\
w &= \epsilon \sum_{n=0}^{\infty} (z_n \sin n\phi) \\
P &= \bar{P} + \epsilon \sum_{n=0}^{\infty} (\eta_n \cos n\phi) \\
\rho &= \bar{\rho} + \epsilon \sum_{n=0}^{\infty} (\xi \cos n\phi)
\end{aligned}
\tag{15}
$$

The boundary conditions of the imaginary flow field are such that upon rotation of the imaginary flow field, the flow must satisfy the requirements that at the yawed cone surface the normal velocity is to be zero, and at the shock wave, the standard Rankine-Hugoniot shock conditions are to be obeyed.

Let the subscript r denote the rotated imaginary flow field, then a point on the surface of the yawed cone in the rotated field is $\theta_{r_s} = \theta_s + \epsilon \cos\phi$. The normal velocity at θ_{r_s} is $v_{r_s} + \epsilon\, w \csc\theta \sin\phi$. Since $\epsilon\, w \approx 0$, then v_{r_s} must be zero. The value of v_{r_s} is also $v_s + v_s' \,\epsilon \cos\phi$, therefore from Eq. (15)

$$
v_{r_s} = \bar{v}_s + \epsilon \sum_{n=0}^{\infty} (y_n \cos n\phi) + \bar{v}_s' \,\epsilon \cos\phi + \epsilon \cos\phi \,\epsilon \sum_{n=0}^{\infty} (y_n' \cos n\phi) = 0.
$$

therefore when $\theta = \theta_s$, the cone surface in the imaginary flow field (since $\bar{v}_s = 0$),

$$\epsilon \sum_{n=0}^{\infty} (y_n \cos n \phi) + \bar{v}_s' \epsilon \cos \phi = 0 \tag{16}$$

neglecting the ϵ^2 terms. Or, equating like coefficients

$$\text{and} \quad \begin{aligned} y_1 &= -\bar{v}_s' \\ y_n &= 0 \quad \text{if } n \neq 1 \end{aligned} \tag{17}$$

Since, from Eqs. (7) and (9), $\bar{v}' = \bar{u}'' = -2\bar{u}$ at θ_s, then at

$$\theta = \theta_s, \quad y_1 = 2 \epsilon \bar{u}_{\theta_s} \tag{18}$$

The shock wave boundary condition is also evaluated in the rotated field. Here a point on the shock wave surface is given by

$$\theta_{r_w} = \theta_w + \epsilon \sum_{0}^{\infty} (\alpha_n \cos n \phi + A_n \sin n \phi)$$

The arguments for this approximation are, as before, that for small changes in ϵ, the change of the shock wave position is a function of ϵ and ϕ. Neglecting ϵ^2 terms and since the function of ϕ is periodic, the above equation is justified. Due to symmetry

$$\theta_{r_w} = \theta_w + \epsilon \sum_{0}^{\infty} \alpha_n \cos n \phi \tag{19}$$

At the shock wave in the rotated field, the Rankine-Hugoniot equations can be given the form

$$\text{at} \quad \theta_{r_w} = \theta_w + \epsilon \sum_0^\infty \alpha_n \cos n\phi \qquad \begin{pmatrix}\text{Continuous}\\ \text{Tangential}\\ \text{Velocities}\end{pmatrix}$$

$$(U\cos\theta_{r_w}, -U\sin\theta_{r_w}, 0)\cdot(1, 0, 0) = (u, v, w)_{\theta_{r_w}}\cdot(1, 0, 0)$$

$$(U\cos\theta_{r_w}, -U\sin\theta_{r_w}, 0)\cdot\left(0, -\csc\theta_{r_w}\cdot\epsilon\sum_0^\infty n\,\alpha_n \sin n\phi, 1\right) =$$

$$(u, v, w)_{\theta_{r_w}}\cdot(1, 0, 0) \tag{20}$$

$$\rho_1 r_1 = \rho_2 r_2 \qquad \text{(continuity)}$$

$$P_2 - P_1 + \rho_1 r_1 (r_2 - r_1) = 0 \qquad \text{(impulse)}$$

$$r_2 = \frac{1}{\gamma+1}\left[\frac{2a_1^2}{r_1} + (\gamma-1) r_1\right] \text{(energy)}$$

where the subscripts 1 and 2 refer to the two sides of the shock wave (side 1 facing the undisturbed uniform air stream), a_1 is the undisturbed sound velocity (so that $a_1^2 = \frac{\gamma P_1}{\rho_1}$), and

$$r_1 = (U\cos\theta_{r_w}, -U\sin\theta_{r_w}, 0)\left(0, 1, \csc\theta_{r_w}\cdot\epsilon\sum_0^\infty n\,\alpha_n \sin\phi\right)$$

$$\begin{Bmatrix}\text{Normal}\\ \text{Velocities}\end{Bmatrix}$$

$$r_2 = (u, v, w)\left(0, 1, \csc\theta_{r_w}\cdot\epsilon\sum_0^\infty n\,\alpha_n \sin n\phi\right)$$

and

$$u = u_{r_w} = u_{\theta_w} + u'_{\theta_w}\,\epsilon\sum_0^\infty \alpha_n \cos n\phi$$

$$v = v_{r_w} = v_{\theta_w} + v'_{\theta_w}\,\epsilon\sum_0^\infty \alpha_n \cos n\phi$$

$$w = w_{r_w} = w_{\theta_w} + w'_{\theta_w} \;\epsilon \sum_{0}^{\infty} \alpha_n \cos n\phi \text{ , etc.}$$

Substituting the values of r_1, r_2, u, v, w, etc. into Eq. (20), the terms that do not vanish with ϵ will cancel, since the non-yaw solution $\bar{u}$, $\bar{v}$, etc. satisfies the same boundary condition (20) with $\epsilon = 0$. The coefficients of $\cos n\phi$ can then be equated, then,

when $\theta = \theta_w$

$$x_n = - \alpha_n (\bar{u}' + U \sin \theta_w)$$

$$z_n \sin \theta_w = n \alpha_n (\bar{v}_{\theta_w} + U \sin \theta_w)$$

$$\alpha_n \left[- \bar{v}_{\theta_w} \cos \theta_w + \bar{v}'_{\theta_w} + \bar{v}_{\theta_w} (\ln \bar{\rho})' \right] + y_n + \xi_n \frac{\bar{v}}{\rho} = 0 \qquad (21)$$

$$\alpha_n \left(\frac{p'}{\bar{\rho}\bar{v}} + \bar{v}' + \bar{v} \cot \theta + 2 U \cos \theta \right)_{\theta_w} + y_n + \frac{\eta_n}{\rho_{\theta_w} \bar{v}_{\theta_w}} = 0$$

$$\alpha_n \left(\bar{v}'_{\theta_w} - \frac{2 a_1^2 \cot \theta_w}{(\gamma + 1) U \sin \theta_w} + \frac{\gamma - 1}{\gamma + 1} U \cos \theta_w \right) + y_n = 0$$

To simplify Eq. (21), the properties given by Eqs. 4, 5, 7 and 9 of the non-yaw solutions were used. Further, since it is easily seen that $\bar{u}'_w + U \sin \theta_w$ cannot be zero, the first of Eq. 21 can be used to eliminate α_n from the others. The resulting simplified equations are

when $\theta = \theta_w$

$$\alpha_n = - \frac{x_n}{\bar{u}' + U \sin \theta_w} \qquad (22)$$

$$mx_n + z_n \sin\theta_w = 0$$

$$2x_n \cot\theta_w + y_n + \xi_n \frac{\bar{u}'}{\rho} = 0$$

$$- x_n \cot\theta_w + y_n + \frac{\eta_n}{\bar{\rho}\,\bar{u}'} = 0 \qquad (22)$$

$$\frac{2x_n}{\gamma + 1}(2\bar{u} - \bar{u}' \cot\theta_w) + y_n(\bar{u}' + U\sin\theta_w) = 0$$

Upon examination of Eqs. 17 and 22, it is seen that the coefficients α_n, x_n, z_n, ξ_n, η_n are functions of y_n and that y_1 is finite but y_n for $n \neq 1$ is zero. Therefore only α_1, x_1, z_1, ξ_1, η_1, y_1 exist. At this point the subscript is dropped.

Thus the solutions to the Eqs. 10 to 14 in the imaginary flow field are

$$u = \bar{u} + \epsilon x \cos\phi$$

$$v = \bar{v} + \epsilon y \cos\phi$$

$$w = \epsilon z \sin\phi \qquad (23)$$

$$P = \bar{P} + \epsilon \eta \cos\phi$$

$$\rho = \bar{\rho} + \epsilon \xi \cos\phi$$

Or in the yawed cone flow field where $\theta_r = \theta + \epsilon\cos\phi$

then $u_r = u + u'\epsilon\cos\phi$

$$v_r = v + v'\epsilon\cos\phi$$

$$w_r = w + w'\epsilon\cos\phi \qquad (24)$$

$$P_r = P + P'\epsilon\cos\phi$$

$$\rho_r = \rho + \rho'\epsilon\cos\phi$$

Now the Eqs. (23) are substituted in Eqs. 10 to 14 in order to evaluate x, and once x is known, the coefficients y, z, η, ξ and α immediately follow from Eqs. 22. The substitution yields (since the terms that do not vanish with ϵ must cancel, for $\bar{u}$, $\bar{v}$, etc., satisfy the equations of motion with $\epsilon = 0$ (or $v = 0$)).

$$y = x' \quad (\text{thus } y' = x'')$$

$$\bar{u}' y' + (\bar{u}'' + \bar{u}) y + \bar{u}' x + \frac{\eta'}{\bar{\rho}} - \xi \frac{\bar{P}'}{\bar{\rho}^2} = 0 \tag{25}$$

$$y' + y \left[\cot\theta + (\ln\bar{\rho})'\right] + 2x + z \csc\theta + \bar{v}\left(\frac{\xi}{\bar{\rho}}\right)' = 0$$

$$\frac{\eta}{\bar{P}} - \frac{\gamma\,\xi}{\bar{\rho}} = d$$

The last expression is obtained from the equation of state (14) which is

$$\ln P + \gamma \ln\rho = f(\epsilon, \phi)$$

Since $\ln\bar{P} - \ln\bar{\rho}$ = constant, following the earlier reasoning for expansion into Fourier's Series, then

$$\ln\frac{P}{\bar{P}} - \gamma\ln\frac{\rho}{\bar{\rho}} = \epsilon\sum_{0}^{\infty} d_n \cos n\phi + D_n \sin\phi$$

$$= \epsilon\, d_1 \cos\phi$$

$$= \epsilon\, d \cos\phi \quad (\text{dropping the subscript})$$

Upon substitution from Eq. (23)

$$\ln\frac{\bar{P} + \epsilon\,\eta\cos\phi}{\bar{P}} - \gamma\ln\frac{\bar{\rho} + \xi\epsilon\cos\phi}{\bar{\rho}} = \epsilon\, d\cos\phi$$

which is equivalent to (neglecting terms of ϵ^2, etc.)

$$\frac{\epsilon\,\eta\,\cos\phi}{\bar{P}} - \frac{\gamma\,\xi\,\epsilon\cos\phi}{\bar{\rho}} = \epsilon\,d\cos\phi$$

and thus

$$\frac{\eta}{\bar{P}} - \frac{\gamma\xi}{\bar{\rho}} = d$$

Equations (25) can now be combined to yield a single equation of x,

$$x'' + x'\left\{\cot\theta + \lambda\left[2\bar{u} + 3\bar{u}\cot\theta + (\gamma+1)\,\lambda\bar{u}'\,(\bar{u} + \bar{u}'\cot\theta)\right]\right\}$$

$$+ x\left\{1 - \cot^2\theta + \lambda\left[-\bar{u}'\cot^2\theta + (\gamma-1)\,\lambda\bar{u}\,(\bar{u} + \bar{u}'\cot\theta)\right]\right\} \tag{26}$$

$$+ \frac{d}{\gamma-1}\sqrt{-\bar{u}'\bar{\rho}\,\sin\theta}\,(1+\lambda\bar{u}')\,\csc^2\theta\cdot\int_{\theta_w}^{\theta}\frac{\bar{P}\,d\theta}{-\bar{u}'\bar{\rho}\sqrt{-\bar{u}'\bar{\rho}\,\sin\theta}} =$$

where

$$\lambda = \frac{2\bar{u}'}{(\gamma-1)(c^2-\bar{u}^2) - (\gamma+1)\,\bar{u}'^2} \tag{27}$$

$$c^2 = U^2 + \frac{2a^2}{\gamma-1}$$

and the boundary conditions 18 and 22 become

when $\theta = \theta_w$

$$x = -\frac{d}{\gamma-1}\,\frac{\bar{P}}{\bar{\rho}}\,\frac{\tan\theta}{\bar{u}' + U\sin\theta}$$

$$x' = \frac{2d}{\gamma^2-1}\cdot\frac{\bar{P}}{\bar{\rho}}\cdot\frac{2U\sin\theta - \bar{u}'}{(\bar{u}' + U\sin\theta)^2} \tag{28}$$

when $\theta = \theta_s$

$$x' = 2\bar{u} \tag{29}$$

For the actual computation, it is preferable to work with nondimensional quantities. Thus Eq. 26 is changed as follows:

$$x'' + B_1 x' + B_2 x + B_3 cd = 0 \tag{30}$$

where

$$B_1 = \cot\theta + \lambda\left[2\bar{u} + \bar{u}'\cot\theta + (\gamma + 1)\lambda\bar{u}'(\bar{u} + \bar{u}'\cot\theta)\right]$$

$$B_2 = 1 - \cot^2\theta + \lambda\left[-\bar{u}'\cot^2\theta + (\gamma - 1)\lambda\bar{u}(\bar{u} + \bar{u}'\cot\theta)\right]$$

$$B_3 = \frac{1}{\gamma - 1}\sqrt{-\bar{u}'\bar{\rho}\sin\theta}\,(1 + \lambda\bar{u}')\csc^2\theta\int_{\theta_w}^{\theta}\frac{\bar{p}\,d\theta}{-\bar{u}'\bar{\rho}\sqrt{-\bar{u}'\bar{\rho}\sin\theta}}$$

Equations 30 and 28 are solved for $\frac{X}{cd}$. Equation 29 next determines d. This gives $\frac{X}{c}$ and $\frac{Y}{c}$ $(= \frac{X'}{c})$, and it is then easy to evaluate the remaining quantities. That is

$$\frac{z}{c} = -\frac{\frac{X}{c}}{\sin\theta} + \frac{d}{\gamma - 1}\frac{\sqrt{-\frac{\bar{u}'}{c}\bar{\rho}\sin\theta}}{\sin\theta}\int_{\theta_w}^{\theta}\frac{\frac{a^2}{c^2}\,d\theta}{-\bar{\rho}\,\gamma\,\frac{\bar{u}'}{c}\sqrt{-\frac{\bar{u}'}{c}\bar{\rho}\sin\theta}}$$

$$\frac{\eta}{p} = -\frac{d}{\gamma - 1} - \frac{\gamma\left(\frac{\bar{u}x}{c^2} + \frac{\bar{v}x'}{c}\right)}{\frac{a^2}{c^2}}$$

$$\frac{\xi}{\rho} = \frac{\frac{\eta}{p} - d}{\gamma}$$

$$\alpha = \frac{\delta}{\epsilon} = -\frac{\frac{x_w}{c}}{\frac{\bar{u}'_w}{c} + \frac{\bar{u}}{c}\sin\theta_w}$$

These equations give the imaginary flow field for a non-yaw cone with finite v-component of velocity caused by an imaginary external force. The boundary conditions at the surfaces of the cone and the shock wave (the position of which is the same as the Taylor and Maccoll solution) are such that upon rotation of these surfaces to coincide with the respective surfaces of the yawed cone in question, the boundary conditions of the yawed motion are satisfied.

The solution to yawing cone motion, then, is as follows (where the subscript, r, denotes the fluid properties about the yawing cone):

Now $\theta_r = \theta + \epsilon \cos \phi$

then $\theta_{r_s} = \theta_s + \epsilon \cos \phi$

$$\theta_{r_w} = \theta_w + \epsilon \alpha \cos \phi$$

$$u_r = u + u' \epsilon \cos \phi = \bar{u} + \epsilon x \cos \phi + \bar{u}' \epsilon \cos \phi$$

$$= \bar{u} + \epsilon x \cos \phi + \bar{v} \quad \epsilon \cos \phi$$

$$v_r = v + v' \epsilon \cos \phi = \bar{v} + \epsilon y \cos \phi + \bar{v}' \epsilon \cos \phi$$

$$w_r = w + w' \epsilon \cos \phi = 0 + \epsilon z \sin \phi + 0$$

$$P_r = P + P' \epsilon \cos \phi = \bar{P} \left(1 + \frac{\eta}{\bar{P}} \epsilon \cos \phi\right) + \bar{P}' \epsilon \cos \phi$$

$$\rho_r = \rho + \rho' \epsilon \cos \phi = \bar{\rho} \left(1 + \frac{\xi}{\bar{\rho}} \epsilon \cos \phi\right) + \bar{\rho}' \epsilon \cos \phi$$

where

$$\bar{v}' = \bar{u}'' = -\frac{(2\bar{u} + \bar{u}' \cot \theta)(c^2 - \bar{u}^2 - \bar{u}'^2) - \frac{2}{\gamma - 1} \bar{u} \bar{u}'^2}{(c^2 - \bar{u}^2 - \bar{u}'^2) - \frac{2}{\gamma - 1} \bar{u}'^2}$$

and

$$\overline{P}' = - \bar{\rho}\,\bar{u}' \,(\bar{u} + \bar{u}'')$$

$$\overline{\rho}' = \frac{\overline{P}'\bar{\rho}}{\gamma\,\overline{P}}$$

Notice that at $\theta_{r_s} = \theta_s + \epsilon \cos \phi$

$$u_{r_s} = \bar{u}_s + \epsilon\, x_s \cos \phi$$

$$v_{r_s} = \epsilon\, y_s \cos \phi + \bar{v}' \;\epsilon \cos \phi = - \epsilon\, 2\, \bar{u}_s \cos \phi + \epsilon\, 2 \bar{u}_s \cos \phi = 0$$

$$w_{r_s} = \epsilon\, z_s \sin \phi$$

$$P_{r_s} = \overline{P}\,(1 + \frac{\eta}{\overline{P}} \;\epsilon \cos \phi)$$

$$\rho_{r_s} = \bar{\rho}\,(1 + \frac{\xi}{\bar{\rho}}\; \epsilon \cos \phi)$$

These equations describe the flow field about a cone yawed at an angle of ϵ with the free stream axis.

Concluding Remarks

The above derivation is intended to elucidate Stone's analysis by simplifying the analytical procedure and describing the basic principles involved. The simplified analysis is necessarily not as rigorous. For the more rigorous analysis and the discussion on the uniqueness of the solution, the original paper should be consulted.

This simplified version of Stone's analysis brings out the point that three systems of fluid flow were analysed and related to yield the solution of the flow field about a slightly yawing cone. The three systems are:

1. Flow about a non-yaw cone with zero w-component of velocity.
2. Flow about a non-yaw cone with finite w-component of velocity.
3. Flow about a yawing cone with finite w-component of velocity.

Acknowledgement

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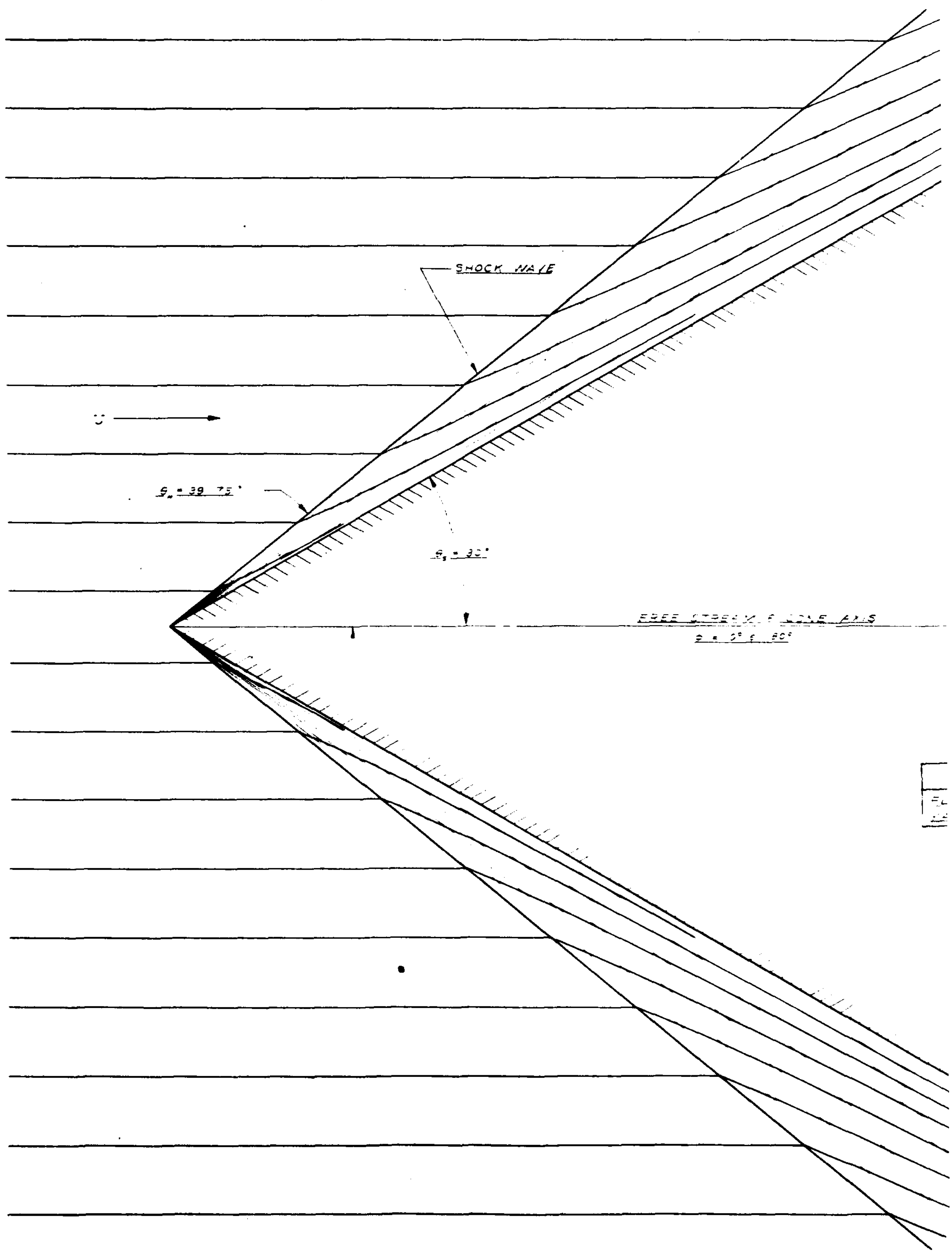

SHOCK WAVE
U
$\theta_w = 39.75°$
$\theta_s = 30°$
FREE STREAM & CONE AXIS
$\phi = 0° \; \& \; 180°$

1

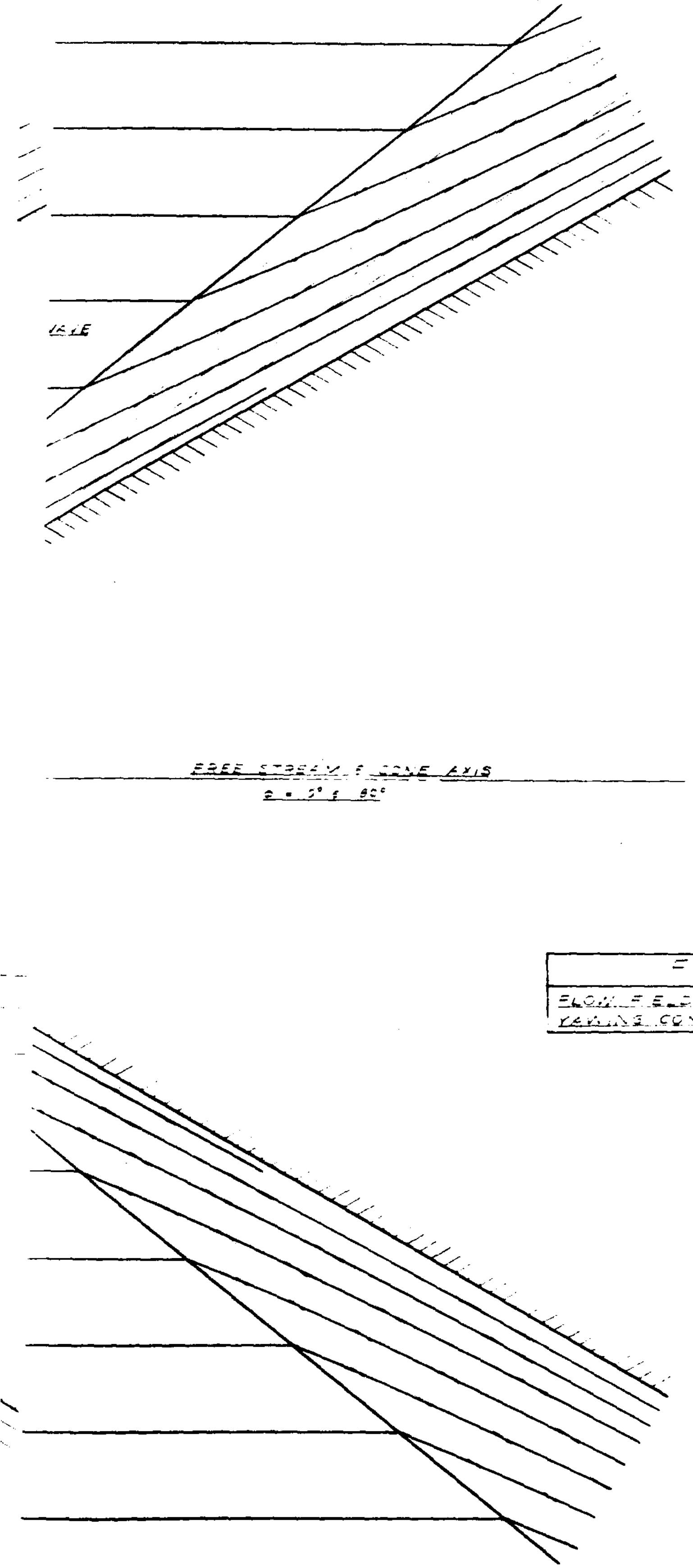
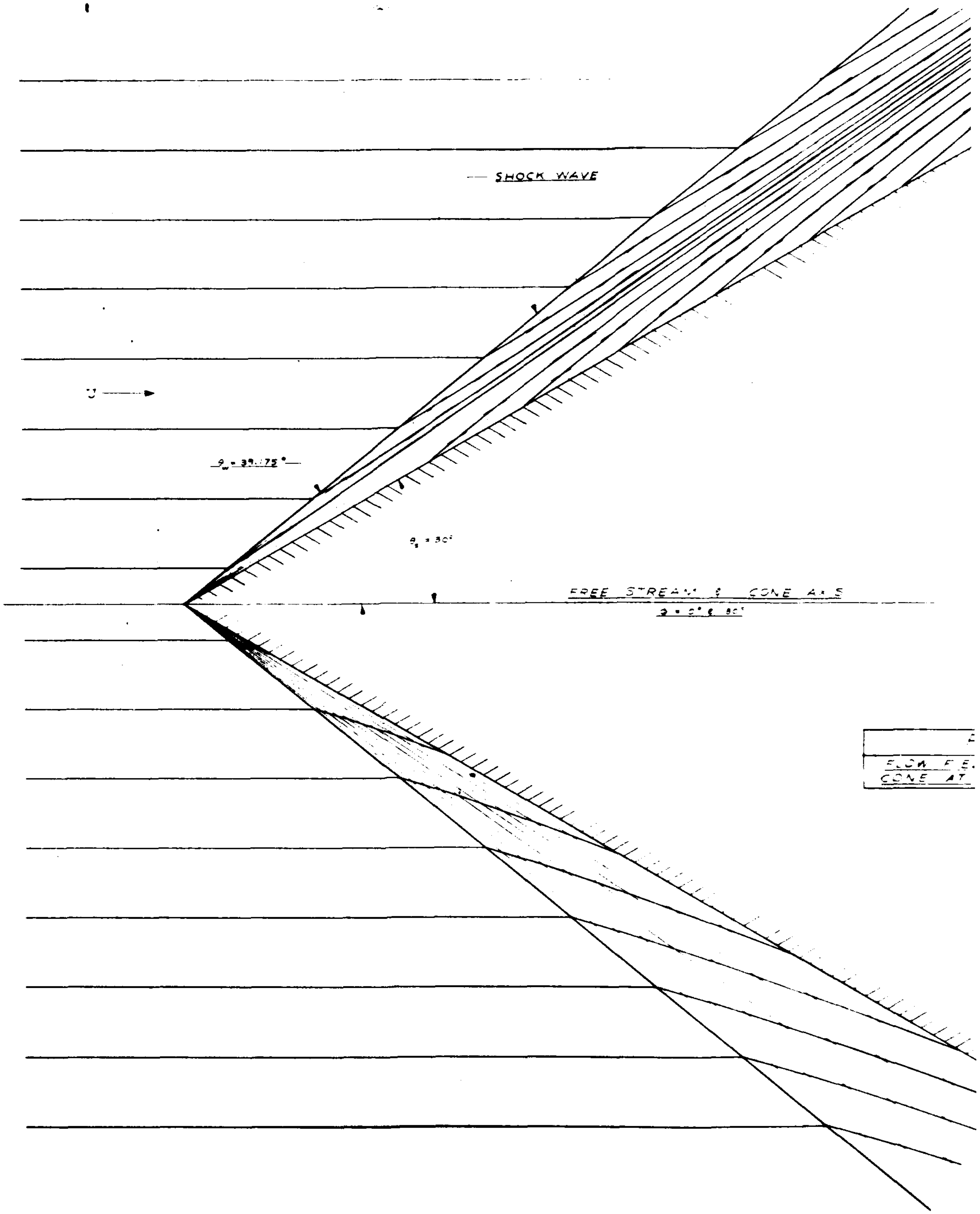


FIG. 1

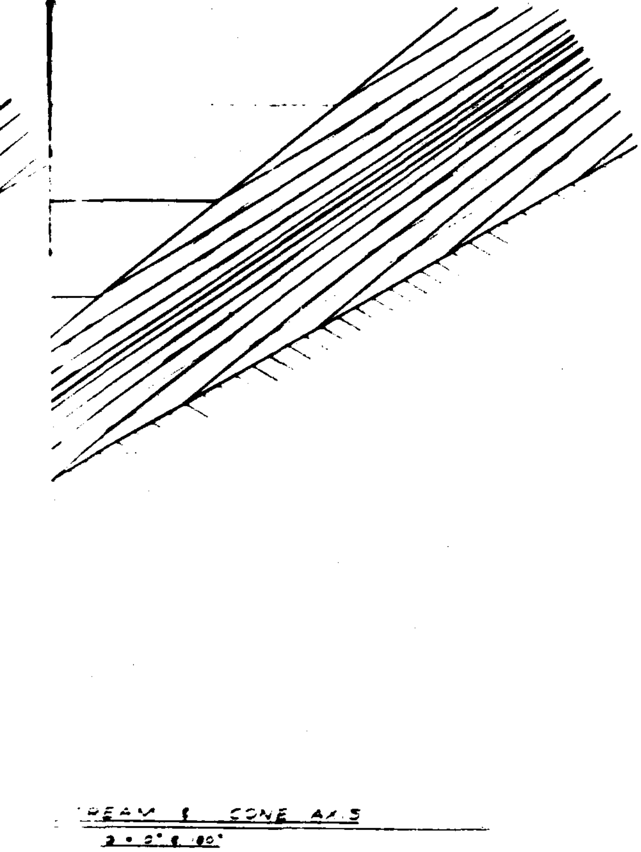
FLOW FIELD ABOUT A [illegible]
YAWING CONE AT [illegible]

2



SHOCK WAVE
U
θw = 39.175°
θs = 30°
FREE STREAM & CONE AXIS
φ = 0° & 90°
FLOW FIE.
CONE AT

1



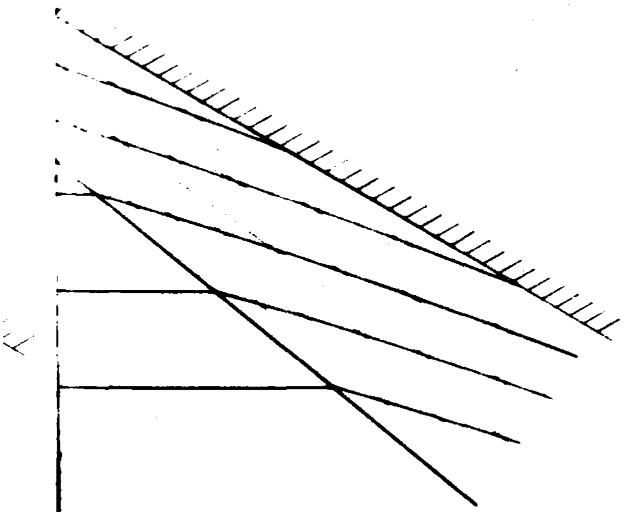
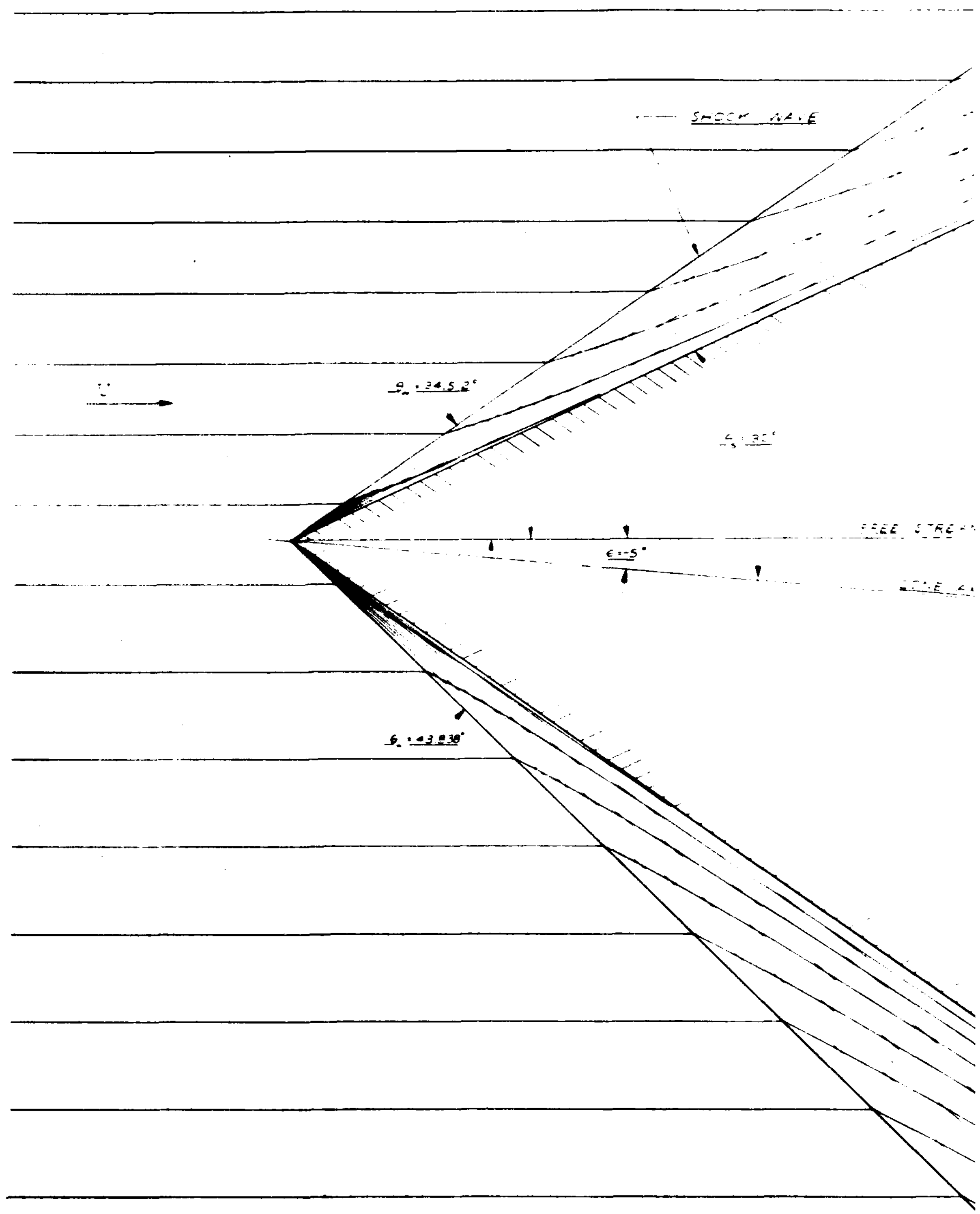


FIG. 2

FLOW FIELD ABOUT A NON-YAWING CONE AT M = 3.1617 [illegible]



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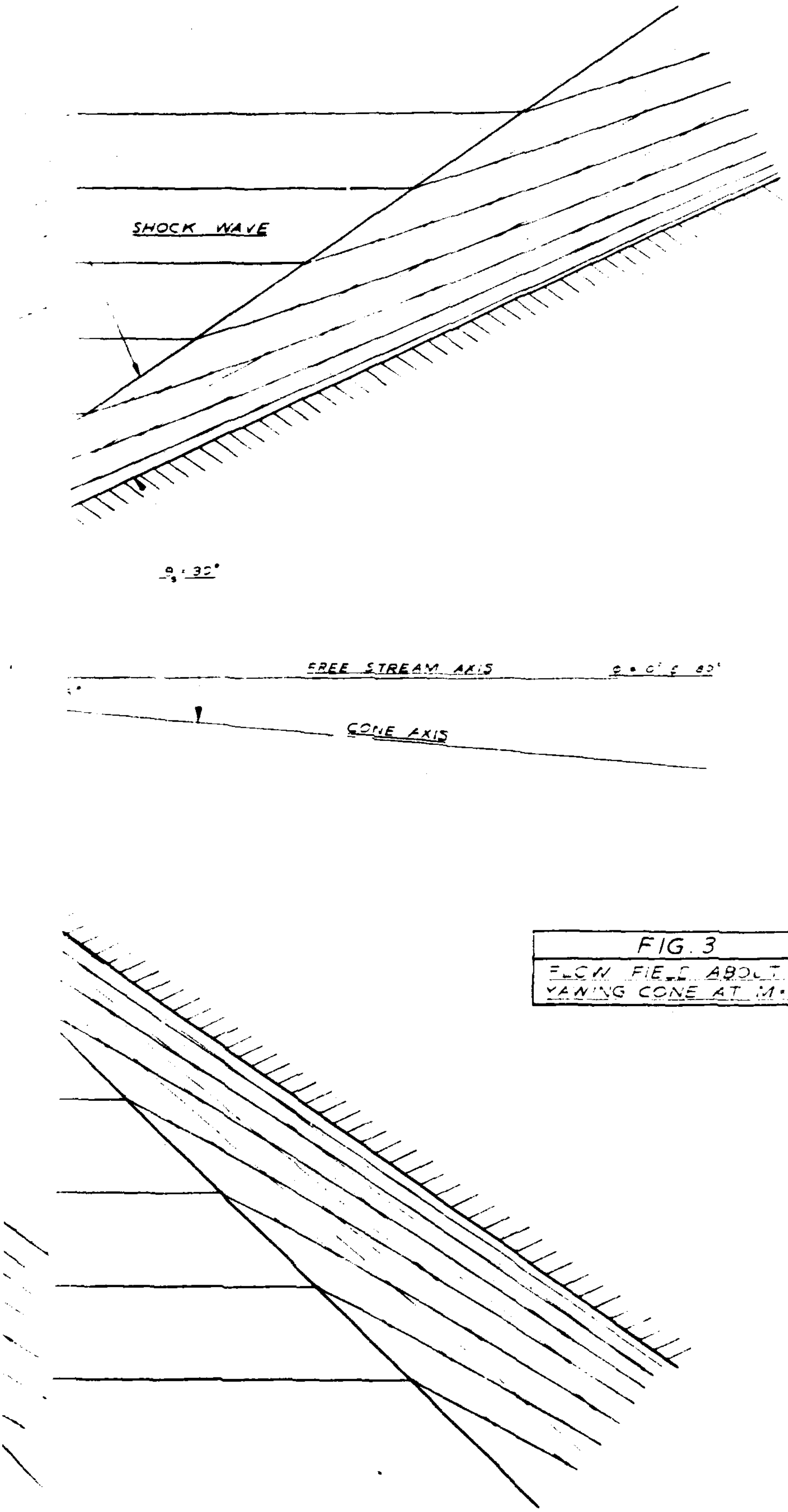

FIG. 3

FLOW FIELD ABOUT A YAWING CONE AT M = 3.617

2